North Sydney Boys High School

MATHEMATICS (EXTENSION 2)

2018 HSC Course Assessment Task 3 (Trial Examination)
Wednesday 27th of June, 2018

General instructions

- Working time – 3 hours. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets + answer sheet used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer sheet provided (numbered as page 13)

SECTION II

- Commence each new question on a new page. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

STUDENT NUMBER: .... **# BOOKLETS USED**: .....

Class (please ✔)

◯ 12M4A – Miss Lee ◯ 12M4B – Dr Jomaa ◯ 12M4C – Mr Ireland

Marker's use only.

QUESTION	1-10	11	12	13	14	15	16	Total
MARKS	$\overline{10}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{15}$	$\overline{100}$

Section I

10 marks
Attempt Question 1 to 10
Allow approximately 15 minutes for this section

Mark your answers on the answer sheet provided.

Questions **Marks**

1. **1**

Let $z = \sqrt{3} + i$. What is the value of $\overline{\left(\dfrac{i}{z}\right)}$?

(A) $1 - i\sqrt{3}$

(B) $\dfrac{1 - i\sqrt{3}}{4}$

(C) $\dfrac{-1 + i\sqrt{3}}{4}$

(D) $\dfrac{\sqrt{3} - i}{4}$

2. An ellipse has foci at $(-5, 0)$ and $(5, 0)$ and its directrices have equations $x = -10$ and $x = 10$. What is the eccentricity of the ellipse? **1**

(A) $\dfrac{1}{\sqrt{2}}$

(B) $\sqrt{2}$

(C) 2

(D) $\dfrac{1}{2}$

3. Given that $3x^3 - 5x + 6 = 0$ has roots α, β and γ. What is the value of $\alpha^3 + \beta^3 + \gamma^3$? **1**

(A) -1

(B) 6

(C) 2

(D) -6

4. Which of the following, for $x > 0$, is an expression for $\displaystyle\int \frac{2}{x + x^3}\, dx$? **1**

(A) $\ln x\sqrt{1 + x^2} + C$

(B) $\ln x^2(1 + x^2) + C$

(C) $\ln \dfrac{x^2}{x^2 + 1} + C$

(D) $\ln \dfrac{x^2}{\sqrt{1 + x^2}} + C$

5. Using a suitable substitution, what is the correct expression for $\displaystyle\int_0^{\frac{\pi}{3}} \sin^4 x \cos^3 x \, dx$ in terms of u? **1**

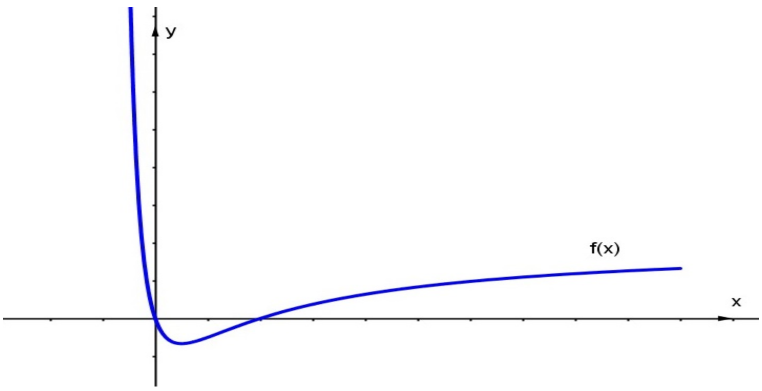
(A) $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \left(u^6 - u^4\right)\, du$

(B) $\displaystyle\int_1^{\frac{1}{2}} \left(u^4 - u^6\right)\, du$

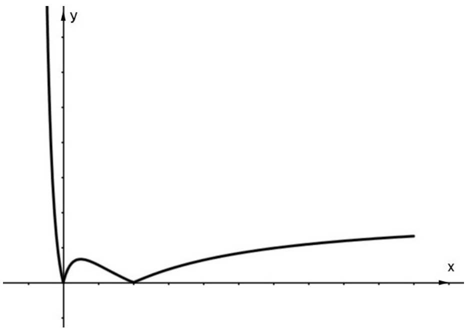
(C) $\displaystyle\int_{\frac{1}{2}}^{1} \left(u^4 - u^6\right)\, du$

(D) $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \left(u^4 - u^6\right)\, du$

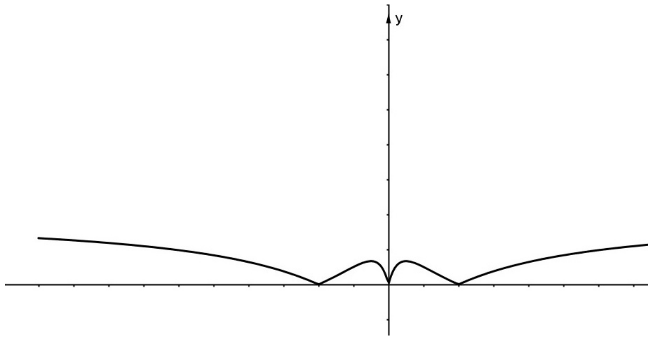
6. Consider the graph of $y = f(x)$ drawn below. **1**



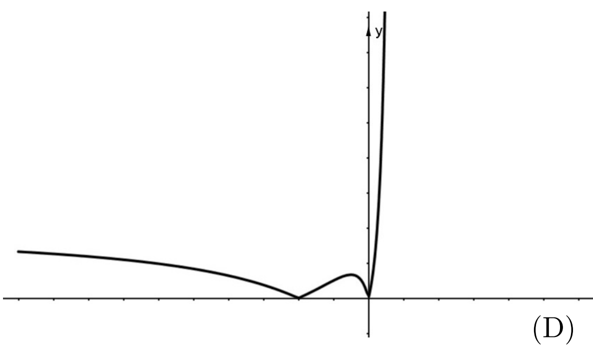
Which of the following diagrams shows the graph of $|f(-x)|$?



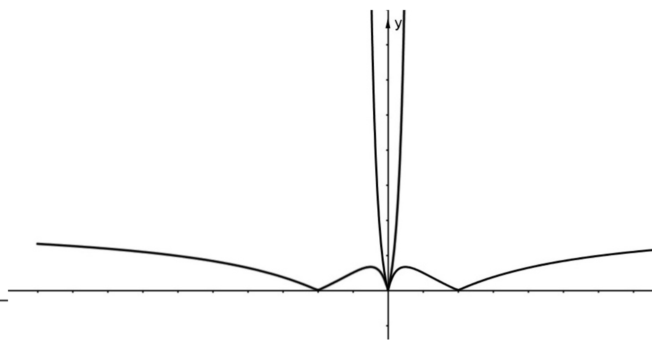
(A)



(C)



(B)



(D)

7. Using implicit differentiation on the equation $y^3 = 3x^2y - 2x^3$, then $\dfrac{dy}{dx}$ would equal **1**

(A) $\dfrac{-2x^2}{y^2 - x^2}$

(B) $\dfrac{2x}{x+y}$

(C) $\dfrac{2x}{x-y}$

(D) $\dfrac{y^2 - x^2}{2x^2}$

8. The normal to the point $P\left(cp, \dfrac{c}{p}\right)$ on the rectangular hyperbola $xy = c^2$ has the equation $p^3x - py + c - cp^4 = 0$. The normal cuts the hyperbola at another point $Q\left(cq, \dfrac{c}{q}\right)$. What is the relationship between p and q? **1**

(A) $pq = -1$

(B) $p^2q = -1$

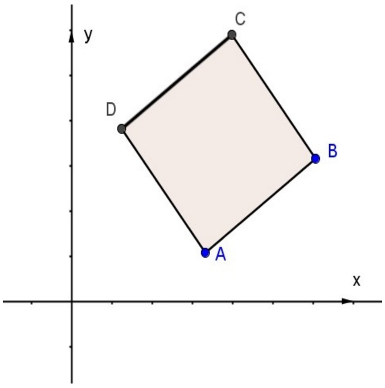
(C) $p^4q = -1$

(D) $p^3q = -1$

9. ω is a non-real root of the equation $z^5 + 1 = 0$. Which of the following is not a root of this equation? **1**

(A) $\bar{\omega}$ (B) ω^2 (C) $\dfrac{1}{\omega}$ (D) ω^3

10. The Argand plane shows the square ABCD in the first quadrant. The point A represents the complex number z and the point C represents the complex number ω. **1**



Which of the following represents the point D?

(A) $\dfrac{z+\omega}{2} + i\dfrac{z-\omega}{2}$

(B) $\dfrac{z-\omega}{2} + i\dfrac{z+\omega}{2}$

(C) $\dfrac{z+\omega}{2} - i\dfrac{z-\omega}{2}$

(D) $\dfrac{z-\omega}{2} - i\dfrac{z+\omega}{2}$

Examination continues overleaf...

Section II

90 marks
Attempt Questions 11 to 16
Allow approximately 2 hours and 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Commence a NEW page. **Marks**

(a) Evaluate $\displaystyle\int \frac{dx}{\sqrt{3+2x-x^2}}$. **2**

(b) Evaluate $\displaystyle\int_{-1}^{0} \frac{2x^3-4x+1}{x-1}\,dx$. **3**

(c) By using $t = \tan\dfrac{x}{2}$, find $\displaystyle\int \frac{dx}{2+\sin x+\cos x}$. **3**

(d) By using integration by parts, find $\displaystyle\int e^{-x}\cos 2x\,dx$. **3**

(e) Consider

$$\frac{36}{(x+4)^2(2-x)} = \frac{a}{x+4} + \frac{b}{(x+4)^2} + \frac{c}{2-x}$$

i. Find a, b and c. **2**

ii. Hence or otherwise, evaluate $\displaystyle\int \frac{36}{(x+4)^2(2-x)}\,dx$. **2**

End of Question 11

Question 12 (15 Marks) Commence a NEW page. **Marks**

(a) Let $z = 1 + i\sqrt{3}$

i. Find the value of $|z|$ **1**

ii. Express $\dfrac{\bar{z}}{z}$ in modulus-argument form. **2**

(b) Sketch the locus of z if $\dfrac{z+3i}{z-3i}$ is purely imaginary. **2**

(c) $1 - i$ is a root of the quadratic equation $z^2 + \omega z - i = 0$. Find the complex number ω in the form $a + ib$. **2**

(d) For the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$, find:

i. The eccentricity. **1**

ii. The coordinates of the foci S and S' and the equations of its directrices. **2**

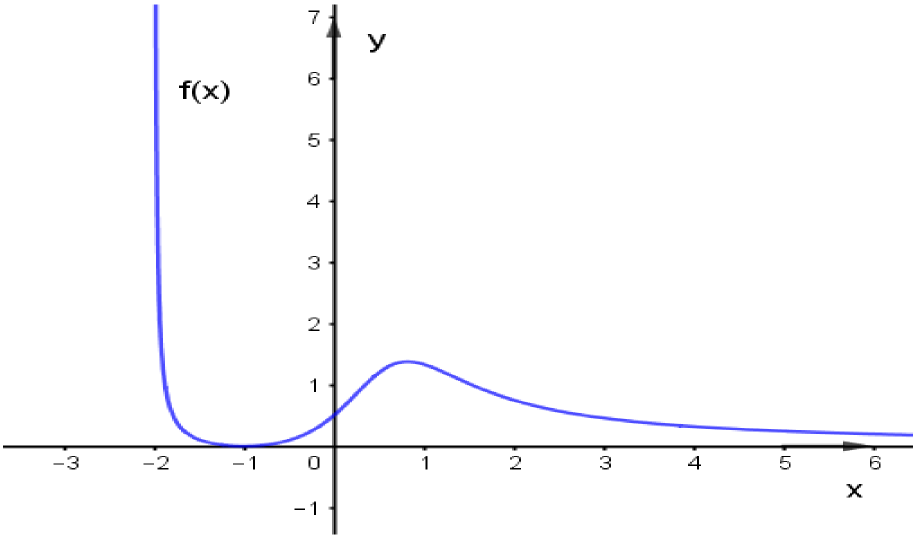
iii. Sketch the ellipse showing all the above features. **2**

(e) The polynomial $x^4 - 3x^3 - 2x^2 + 2x + 1 = 0$ has roots α, β and γ. Find an equation with roots $\alpha^2 - 1$, $\beta^2 - 1$ and $\gamma^2 - 1$. **3**

End of Question 12

Question 13 (15 Marks) Commence a NEW page. **Marks**

(a) The diagram shows the graph of the function $y = f(x)$.



Draw separate one-third page sketches of graphs of the following:

i. $y = \sqrt{f(x)}$. **2**

ii. $y = \dfrac{1}{f(x)}$. **2**

iii. $y = xf(x)$. **2**

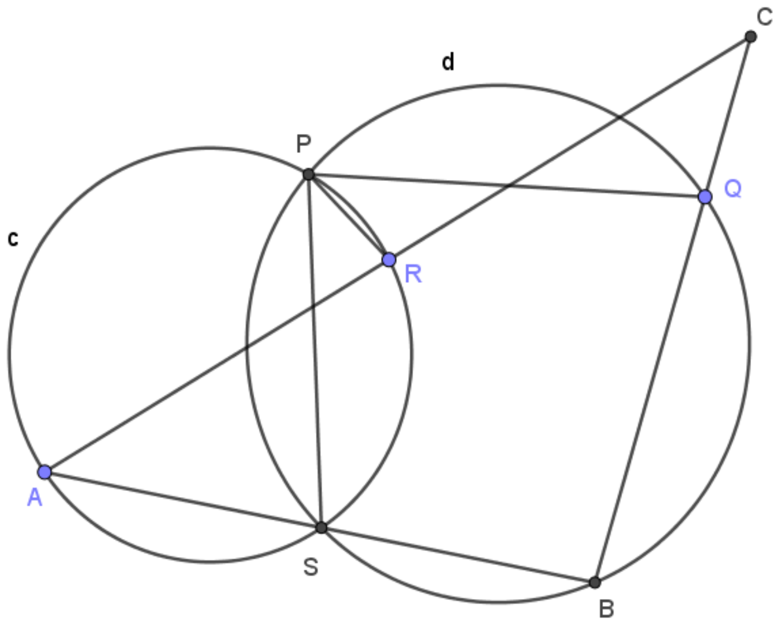
(b) A solid is formed by rotating the area enclosed by the curve $x^2 + y^2 = 9$ through one complete revolution about the line $x = 7$.

i. By taking slices perpendicular to the axis of rotation, show that the volume of the solid is $V = 28\pi \displaystyle\int_{-3}^{3} \sqrt{9 - y^2}\,dy$ **3**

ii. Find the exact volume of the solid. **2**

Question 13 continues on the next page...

(c) Two circles c and d meet at P and S. Points A and R lie on c and points B and Q lie on d. AB passes through S and AR produced meets BQ produced at C, as shown in the diagram.



i. **Copy the diagram to your booklet.**

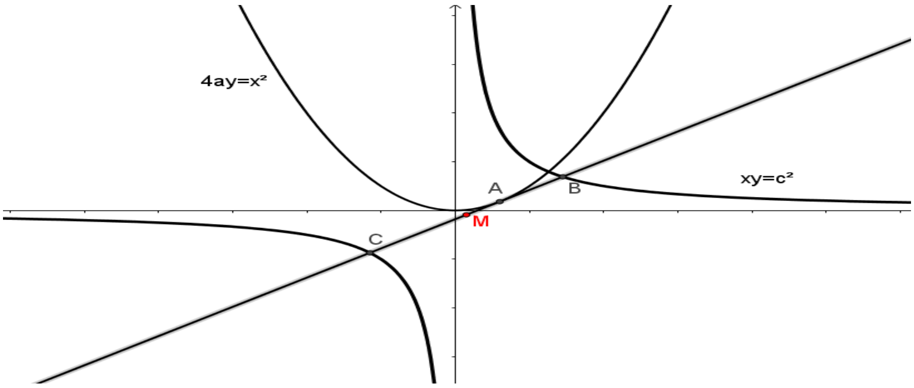
ii. Prove that $\angle PRA = \angle PQB$. **2**

iii. Prove that the points P, R, Q and C are concyclic. **2**

End of Question 13

Question 14 (15 Marks) Commence a NEW page. **Marks**

(a) Consider the hyperbola $xy = c^2$ and the parabola $4ay = x^2$. Let $A(2at, at^2)$ lie on the parabola.

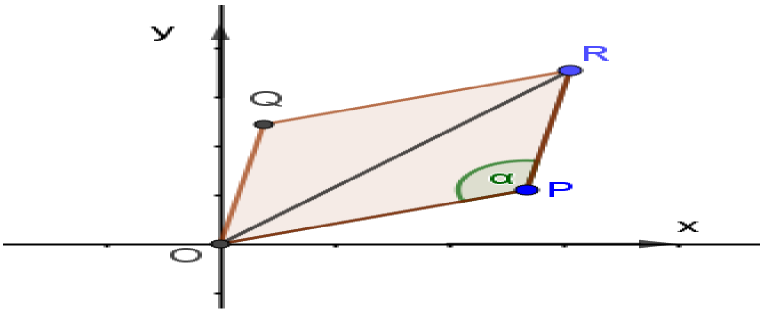


i. Derive the equation of the tangent at A. **2**

ii. The tangent in (i) cuts the hyperbola at B and C. Without finding the coordinates of B and C, find the coordinates of M the midpoint of BC. **3**

iii. Hence, find the equation of the locus of M as A moves on the parabola, stating all restrictions. **2**

(b) The diagram shows the parallelogram OQRP in the Argand plane with the point P represented by the complex number z and Q represented by the complex number ω and $\alpha = \angle OPR$.



i. Show that $\arg(\dfrac{\omega}{z}) = \pi - \alpha$ **1**

ii. Use the cosine rule to show that **2**

$$|z+\omega|^2 = |z|^2 + |\omega|^2 + 2\,|z|\,|\omega| \cos\left(\arg\left(\frac{\omega}{z}\right)\right)$$

iii. Hence or otherwise show that **2**

$$\cos\left(\arg\left(\frac{\omega}{z}\right)\right) = \frac{|z+\omega|^2 - |\omega - z|^2}{4\,|\omega|\,|z|}$$

iv. If $\alpha = \dfrac{7\pi}{12}$, $|PR| = 2$ and $z = 3(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4})$, find in Cartesian form the complex number $\omega + z$. **3**

End of Question 14

Question 15 (15 Marks) Commence a NEW page. **Marks**

(a) Given that $z = \cos\theta + i\sin\theta$ and $z^n - \dfrac{1}{z^n} = 2i\sin n\theta$.

i. By considering $\left(z - \dfrac{1}{z}\right)^5$, show that **2**

$$\sin^5\theta = \frac{1}{16}\left(\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right).$$

ii. Solve the following equation for $0 \le \theta \le 2\pi$ **3**

$$\sin 5\theta - 5\sin 3\theta + 6\sin\theta = 0.$$

(b) If $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \sec^n x dx$

i. Show that $I_n = \dfrac{\left(\sqrt{2}\right)^{n-2}}{n-1} + \dfrac{n-2}{n-1} I_{n-2}, \quad \text{for} \quad n \ge 2.$ **3**

ii. Hence or otherwise, evaluate $\displaystyle\int_0^1 \left(1 + x^2\right)^{\frac{5}{2}} dx$. **3**

(c) Let a and b be real numbers. Consider the cubic equation

$$x^3 - 2bx^2 - a^2x + b^2 = 0$$

i. Show that if $x = -1$ is a solution, then $1 - \sqrt{2} \le b \le 1 + \sqrt{2}$. **2**

ii. Show that there is no value of b for which $x = -1$ is a repeated root. **2**

End of Question 15

Question 16 (15 Marks) Commence a NEW page. **Marks**

(a) Suppose that x is apositive real number.

i. Find the sum of the geometric series **1**

$$1 - t^3 + t^6 - t^9 + \cdots + t^{6n}$$

ii. Hence, show that **1**

$$\frac{1}{1+t^3} < 1 - t^3 + t^6 - t^9 + \cdots + t^{6n}, \quad \text{for} \quad 0 < t < x.$$

iii. Find the sum of the geometric series **1**

$$1 - t^3 + t^6 - t^9 + \cdots + t^{6n} - t^{6n+3}$$

iv. Hence, show that **1**

$$1 - t^3 + t^6 - t^9 + \cdots + t^{6n} < \frac{1}{1+t^3} + t^{6n+3}, \quad \text{for} \quad 0 < t < x.$$

v. Multiply the inequalities of part (ii) and (iv) by a suitable factor, then integrating from $t = 0$ to $t = x$, show that **3**

$$\frac{1}{3}\ln(1+x^3) < \frac{x^3}{3} - \frac{x^6}{6} + \cdots + \frac{x^{6n+3}}{6n+3} < \frac{1}{3}\ln(1+x^3) + \frac{x^{6(n+1)}}{6(n+1)}.$$

vi. By taking limit as $n \to \infty$, show that for $0 \le x \le 1$ **1**

$$\ln(1+x^3) = x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \ldots$$

vii. Use suitable substitution to prove that **1**

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

viii. Hence or otherwise prove that **2**

$$\ln 4 = 1 + \frac{1}{2.3} + \frac{1}{3.5} + \frac{1}{4.7} + \ldots$$

(b) Given that a, b and c are all positives such that $a+b+c = 1$ and $a+b+c \ge 3\sqrt[3]{abc}$.

i. If x, y and z are all positives, show that **1**

$$\frac{1}{xy} + x + y \ge 3$$

ii. Hence or otherwise, prove that **3**

$$\frac{1}{a(a+1)} + \frac{1}{b(b+1)} + \frac{1}{c(c+1)} \ge 4$$

End of paper.

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

STUDENT NUMBER: ....

Class (please ✔)

◯ 12M4A – Miss Lee ◯ 12M4B – Dr Jomaa ◯ 12M4C – Mr Lin

1 – Ⓐ Ⓑ Ⓒ Ⓓ

2 – Ⓐ Ⓑ Ⓒ Ⓓ

3 – Ⓐ Ⓑ Ⓒ Ⓓ

4 – Ⓐ Ⓑ Ⓒ Ⓓ

5 – Ⓐ Ⓑ Ⓒ Ⓓ

6 – Ⓐ Ⓑ Ⓒ Ⓓ

7 – Ⓐ Ⓑ Ⓒ Ⓓ

8 – Ⓐ Ⓑ Ⓒ Ⓓ

9 – Ⓐ Ⓑ Ⓒ Ⓓ

10 – Ⓐ Ⓑ Ⓒ Ⓓ

MC:

1. $z = \sqrt{3} + i$

$$\overline{\frac{i}{z}} = \frac{\overline{i}}{\overline{z}} = \frac{\overline{i}z}{|z|^2} = \frac{-iz}{|z|^2} = \frac{-i(\sqrt{3}+i)}{4} = \frac{1-i\sqrt{3}}{4}$$ (B)

2. $ae = 5$ and $\frac{a}{e} = 10$

$10e^2 = 5 \therefore e^2 = \frac{1}{2} \therefore e = \frac{1}{\sqrt{2}}$ (A)

3. $$\alpha^3 + \beta^3 + \gamma^3 = \frac{5(\alpha+\beta+\gamma) - 3 \times 6}{3} = -6$$ (D)

4. $$\frac{2}{x+x^3} = \frac{A}{x} + \frac{Bx+C}{1+x^2} = \frac{A(1+x^2)+(Bx+C)x}{x(1+x^2)} = \frac{Ax^2+A+Bx^2+Cx}{x(1+x^2)}$$

$A+B=0$, $C=0$, $A=2$ $\therefore$ $B=-2$

$$\frac{2}{x+x^3} = \frac{2}{x} - \frac{2x}{1+x^2}$$

$$\int \frac{2\,dx}{x+x^3} = 2\ln(x) - \ln|1+x^2| + c = \ln\left|\frac{x^2}{1+x^2}\right| + c$$ (C)

5. $$\int_0^{\pi/3} \sin^4 x\,(1-\sin^2 x)\cos x\,dx = \int_0^{\pi/3} (\sin^4 x - \sin^6 x)\cos x\,dx$$

let $u = \sin x$ $\quad du = \cos x\,dx$

$u = 0 \quad (x=0)$

$u = \frac{\sqrt{3}}{2} \quad (x = \pi/3)$

$$= \int_0^{\sqrt{3}/2} (u^4 - u^6)\,du$$ (D)

6 (B)

7 $$3y^2\frac{dy}{dx} = 3x^2\frac{dy}{dx} + 6xy - 6x^2$$

$$3(y^2-x^2)\frac{dy}{dx} = 6x(y-x)$$

$$\frac{dy}{dx} = \frac{2x}{y+x}$$ (B)

8. $p^3x - py + c - cp^4 = 0$

$p^3cq - \frac{pc}{q} + c - cp^4 = 0$

$cp^3q^2 - cp + cq - cp^4q = 0$

$p^3q^2 - p^4q = p - q$

$p^3q(q-p) = p - q \quad \therefore \boxed{p^3q = -1}$ D

9. B

10.

$\overrightarrow{AD} = i\,\overrightarrow{AB}$

$= i\,\overrightarrow{DC}$

$z_D - z = i(w - z_D)$

$z_D(1+i) = z + iw$

$z_D = \frac{z}{1+i} + \frac{iw}{1+i}$

$= \frac{z(1-i)}{2} + \frac{i(1-i)w}{2}$

$= \frac{z}{2} - i\frac{z}{2} + \frac{i}{2}w + \frac{w}{2}$

$= \frac{z+w}{2} + \frac{i(z-w)}{2}$ C

Question 11:

(a) $$\int \frac{dx}{\sqrt{3+2x-x^2}} = \int \frac{dx}{\sqrt{4-(x-1)^2}} = \boxed{\sin^{-1}\left(\frac{x-1}{2}\right) + C}$$

(b) $$\int_{-1}^{0} \frac{2x^3-4x+1}{x-1}\,dx = \int_{-1}^{0} \frac{2x^3-2x-2x+2-1}{x-1}\,dx$$

$$= 2\int_{-1}^{0} \frac{x(x^2-1)}{x-1}\,dx - 2\int_{-1}^{0} \frac{x-1}{x-1}\,dx - \int_{-1}^{0} \frac{dx}{x-1}$$

$$= 2\int_{-1}^{0} x(x+1)\,dx - 2\int_{-1}^{0} dx - \int_{-1}^{0} \frac{dx}{x-1}$$

$$= 2\left[\frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{0} - 2\left[x\right]_{-1}^{0} - \ln|x-1|\Big|_{-1}^{0}$$

$$= 2\left(0 - \left(\frac{-1}{3}+\frac{1}{2}\right)\right) - 2\left(0-(-1)\right) - \left(\ln 1 - \ln 2\right)$$

$$= \boxed{\frac{-1}{3} - 2 + \ln 2}$$

(c) $t = \tan\frac{x}{2}$, $dt = \frac{1}{2}(1+t^2)\,dx \therefore dx = \frac{2\,dt}{1+t^2}$

$$2 + \sin x + \cos x = 2 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = \frac{2+2t^2+2t+1-t^2}{1+t^2} = \frac{t^2+2t+3}{1+t^2}$$

$$\frac{1}{2+\sin x+\cos x} = \frac{1+t^2}{t^2+2t+3}$$

$$\int \frac{1}{2+\sin x+\cos x}\,dx = \int \frac{1+t^2}{t^2+2t+3} \times \frac{2\,dt}{1+t^2} = 2\int \frac{dt}{(t+1)^2+2}$$

$$= 2 \times \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t+1}{\sqrt{2}}\right) + c$$

$$= \boxed{\sqrt{2}\tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{\sqrt{2}}\right) + C}$$

(d) $I = \int e^{-x}\cos 2x\,dx$

let $u = \cos 2x \therefore du = -2\sin 2x\,dx$

$e^{-x}dx = dv \therefore v = -e^{-x}$

$I = uv - \int v\,du$

$= -e^{-x}\cos 2x - 2\int e^{-x}\sin 2x\,dx$ ✓

let $u = \sin 2x \therefore du = 2\cos 2x\,dx$

$e^{-x}dx = dv \therefore v = -e^{-x}$

$I = -e^{-x}\cos 2x - 2\left[-e^{-x}\sin 2x + 2\int e^{-x}\cos 2x\,dx\right]$ ✓

$= -e^{-x}\cos 2x + 2e^{-x}\sin 2x - 4I$

$5I = e^{-x}(2\sin 2x - \cos 2x)$

$\boxed{I = \frac{e^{-x}}{5}(2\sin 2x - \cos 2x)}$ ✓

(e) $\frac{36}{(x+4)^2(2-x)} = \frac{a}{x+4} + \frac{b}{(x+4)^2} + \frac{c}{2-x}$

(i) multiply by $(x+4)^2$ then set $x=-4$

$\frac{36}{6} = b \therefore \boxed{b = 6}$

multiply by $2-x$ then set $x = 2$ ✓

$\frac{36}{36} = c \therefore \boxed{c = 1}$

Set $x = 0 \therefore \frac{36}{16\times 2} = \frac{a}{4} + \frac{b}{16} + \frac{c}{2}$

$3 = a + \frac{b}{4} + 2c$

$a = 1 - \frac{3}{2} = \boxed{\frac{-1}{2}}$ ✓

(ii) $\displaystyle\int \frac{36}{(x+4)^2(2-x)}\,dx = -\frac{1}{2}\int \frac{dx}{x+4} + 6\int \frac{dx}{(x+4)^2} + \int \frac{dx}{2-x}$ ✓

$$= \boxed{-\frac{1}{2}\ln|x+4| - \frac{6}{x+4} - \ln|2-x| + C}$$ ✓

Question 12

(a) $z = 1 + i\sqrt{3}$

(i) $|z| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

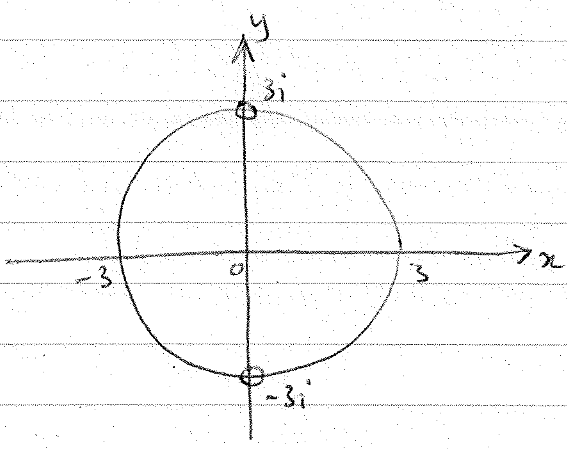
(ii) $\dfrac{\bar{z}}{z} = \dfrac{\bar{z}}{z} \times \dfrac{\bar{z}}{\bar{z}} = \dfrac{(\bar{z})^2}{|z|^2} = \dfrac{(1 - i\sqrt{3})^2}{4} = \left(\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}\right)^2 = \left[\operatorname{cis}(-\pi/3)\right]^2$

$\dfrac{\bar{z}}{z} = \operatorname{cis}\left(\dfrac{-2\pi}{3}\right) = \cos\left(\dfrac{-2\pi}{3}\right) + i\sin\left(\dfrac{-2\pi}{3}\right)$.

(b) $\dfrac{z+3i}{z-3i}$ is purely imaginary

$\therefore \arg\left(\dfrac{z+3i}{z-3i}\right) = \pm\pi/2$

so the locus of z is the circle $x^2 + y^2 = 9$ excluding $(0,3)$ and $(0,-3)$.



Algebraically, let $z = x + iy$

$\dfrac{z+3i}{z-3i} = \dfrac{x + i(y+3)}{x + i(y-3)}$

$= \dfrac{(x + i(y+3))(x - i(y-3))}{x^2 + (y-3)^2}$

$\dfrac{z+3i}{z-3i} = \dfrac{x^2 + (y+3)(y-3) + i(xy + 3x - yx + 3x)}{x^2 + (y-3)^2}$

$= \dfrac{x^2 + y^2 - 9 + 6xi}{x^2 + (y-3)^2}$

$\dfrac{z+3i}{z-3i}$ is purely imaginary $\therefore \operatorname{Re}\left(\dfrac{z+3i}{z-3i}\right) = 0 \therefore x^2 + y^2 - 9 = 0$

$\therefore \boxed{x^2 + y^2 = 9}$ excluding $z = \pm 3i$

(c) $(1-i)^2 + w(1-i) - i = 0$

$-2i + w(1-i) - i = 0 \quad \therefore w = \dfrac{3i}{1-i} = \dfrac{3i(1+i)}{2}$

$\boxed{w = \dfrac{-3}{2} + \dfrac{3}{2}i}$

d)

$$a = 3, b = 4.$$

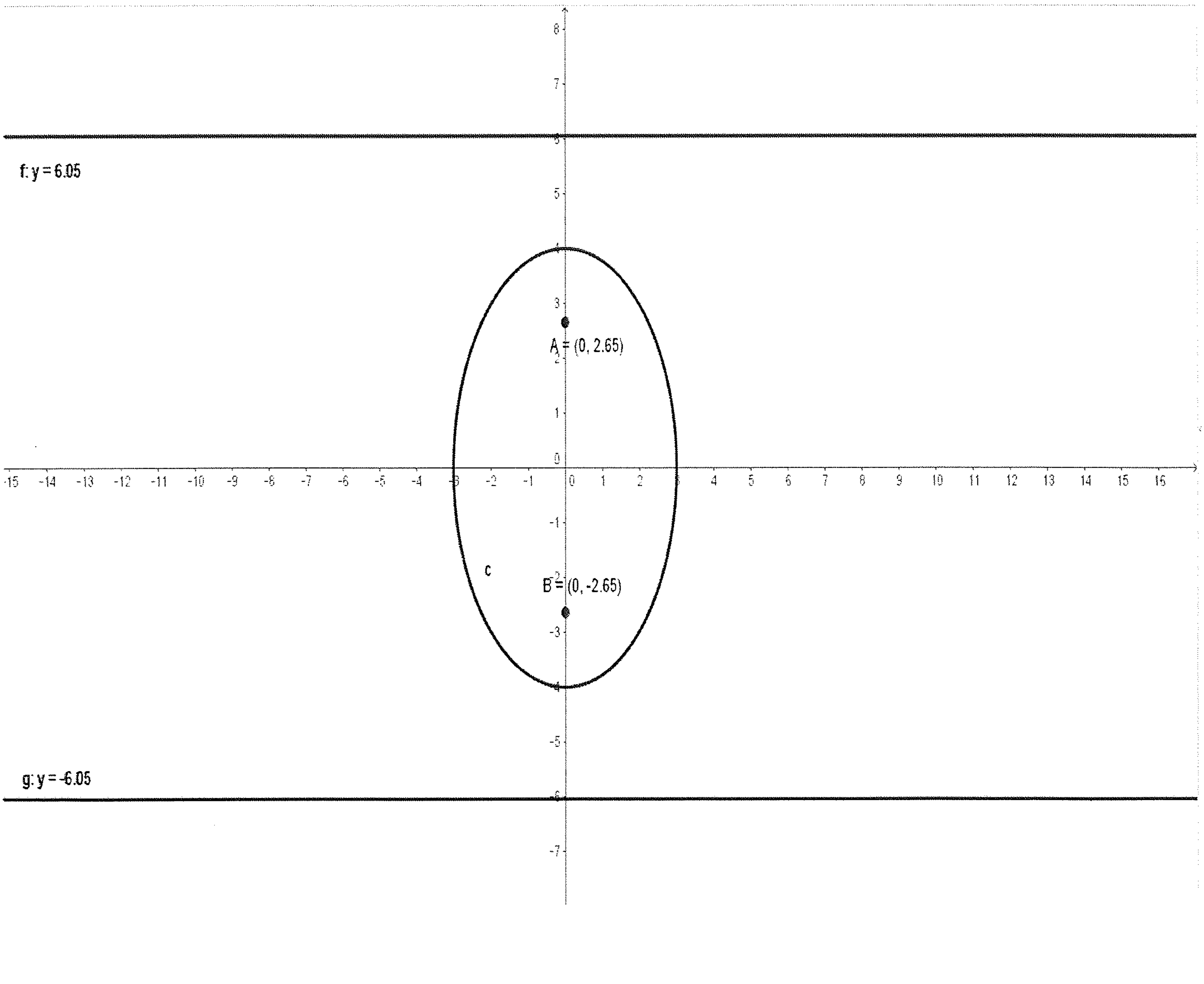
i. $$a^2 = b^2 * (1 - e^2)$$

$$e = \sqrt{1 - \frac{a^2}{b^2}} = \frac{\sqrt{7}}{4}$$

ii. Foci: S(0,$\pm\sqrt{7}$)

Directices: $y = \pm\frac{16\sqrt{7}}{7}$

iii.



(e) Let $\alpha^2 - 1 = y \therefore \alpha^2 = y + 1$

$$\alpha^4 = (y+1)^2$$
$$\alpha^3 = (y+1)\sqrt{y+1}$$
$$\alpha = \sqrt{y+1}$$

α is a root of $x^4 - 3x^3 - 2x^2 + 2x + 1 = 0$

$$\therefore (y+1)^2 - 3(y+1)\sqrt{y+1} - 2(y+1) + 2\sqrt{y+1} + 1 = 0$$

$$(y+1)^2 - 2(y+1) + 1 = 3(y+1)\sqrt{y+1} - 2\sqrt{y+1}$$

$$(y+1-1)^2 = (3y+3-2)\sqrt{y+1}$$

$$y^2 = (3y+1)\sqrt{y+1}$$

$$(y^2)^2 = (3y+1)^2(\sqrt{y+1})^2$$

$$y^4 = (9y^2+6y+1)(y+1)$$
$$= 9y^3 + 9y^2 + 6y^2 + 6y + y + 1$$
$$= 9y^3 + 15y^2 + 7y + 1$$

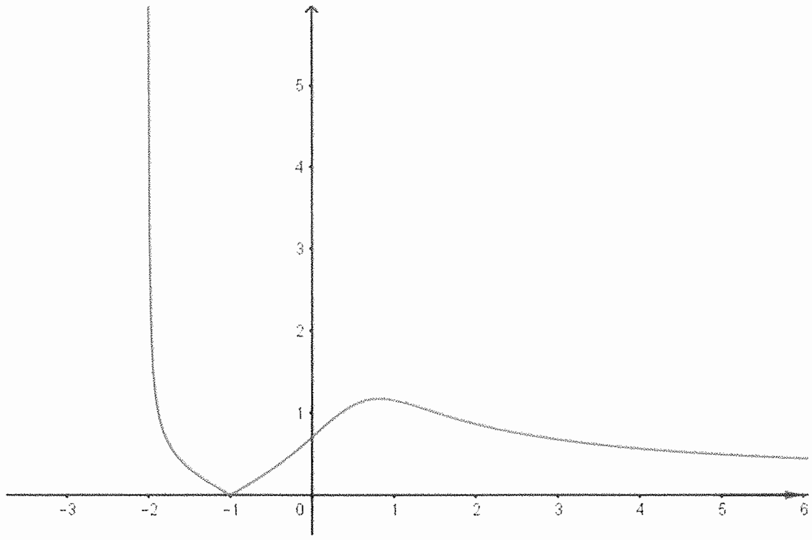
$$\therefore y^4 - 9y^3 - 15y^2 - 7y - 1 = 0$$

$\therefore$ The equation which has roots $\alpha^2-1, \beta^2-1, \gamma^2-1, \delta^2-1$ is

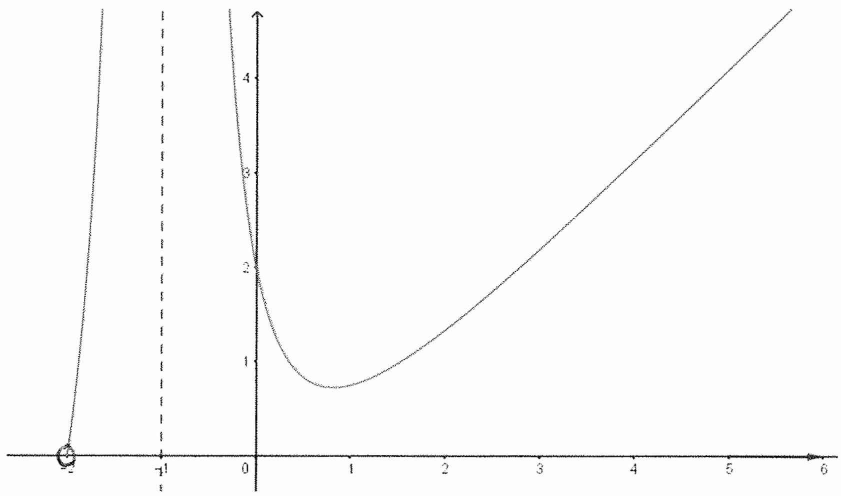
$$\boxed{x^4 - 9x^3 - 15x^2 - 7x - 1 = 0}$$

Question 13
(a)

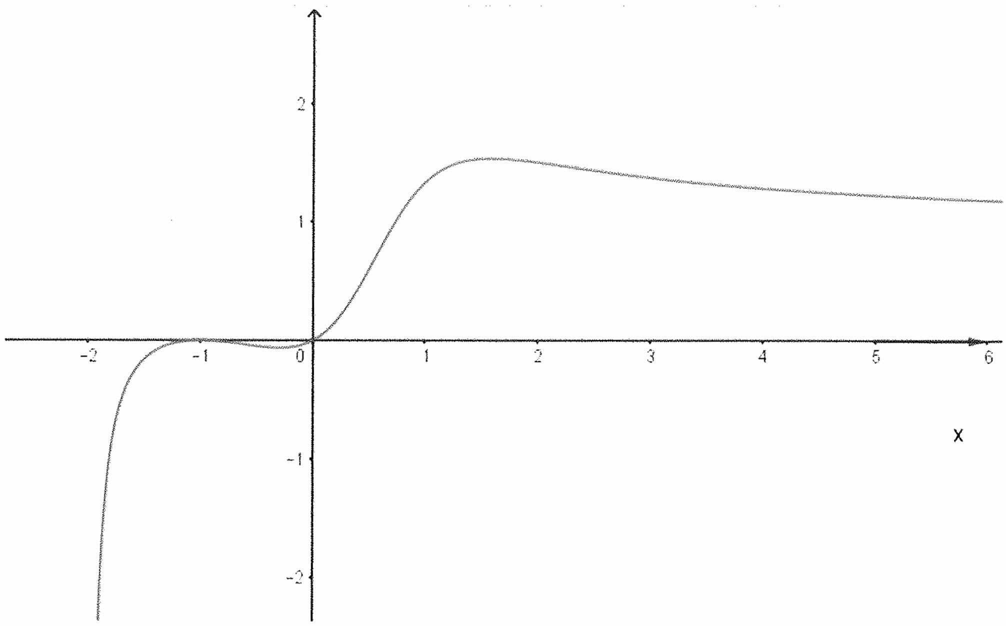
(i) $y = \sqrt{f(x)}$



(ii) $y = \frac{1}{f(x)}$

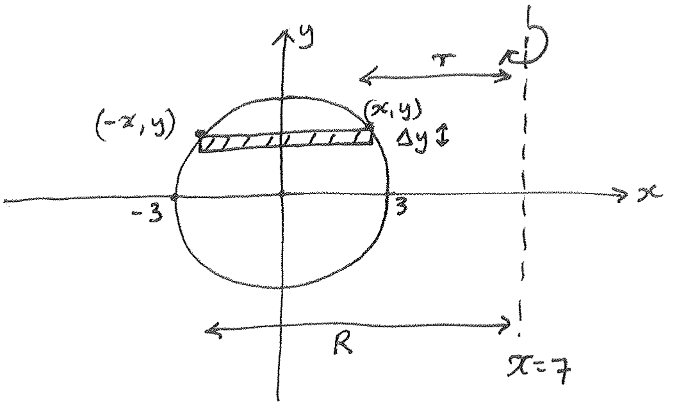


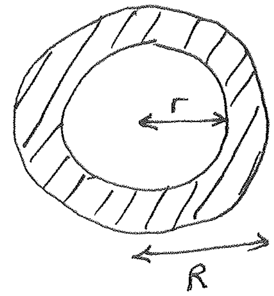
(iii) $y = xf(x)$



13(b)

(i)





$r = 7 - x = 7 - \sqrt{9-y^2}$

$R = 7 + x = 7 + \sqrt{9-y^2}$

(R and r are the outer & inner radii when the hatched strip is rotated about line $x = 7$.)

Area of annular cross section is

$$\Delta A = \pi R^2 - \pi r^2 = \pi (R+r)(R-r)$$
$$= \pi (14)(2\sqrt{9-y^2})$$

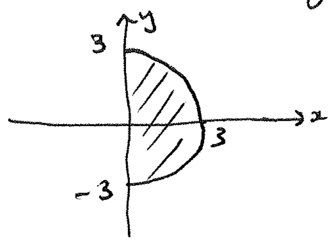
$$\therefore \Delta A = 28\pi \sqrt{9-y^2}$$

Let slice thickness be Δy. Then

Volume element is $\Delta V = 28\pi\sqrt{9-y^2} \cdot \Delta y$

$$\therefore V = \lim_{\Delta y \to 0} \sum_{-3}^{3} 28\pi\sqrt{9-y^2} \cdot \Delta y = 28\pi \int_{-3}^{3} \sqrt{9-y^2}\, dy.$$

(ii) $\int_{-3}^{3} \sqrt{9-y^2}\, dy$ is the area of a semicircle:



$\therefore$ it equals $\frac{1}{2} \times \pi \times 3^2$

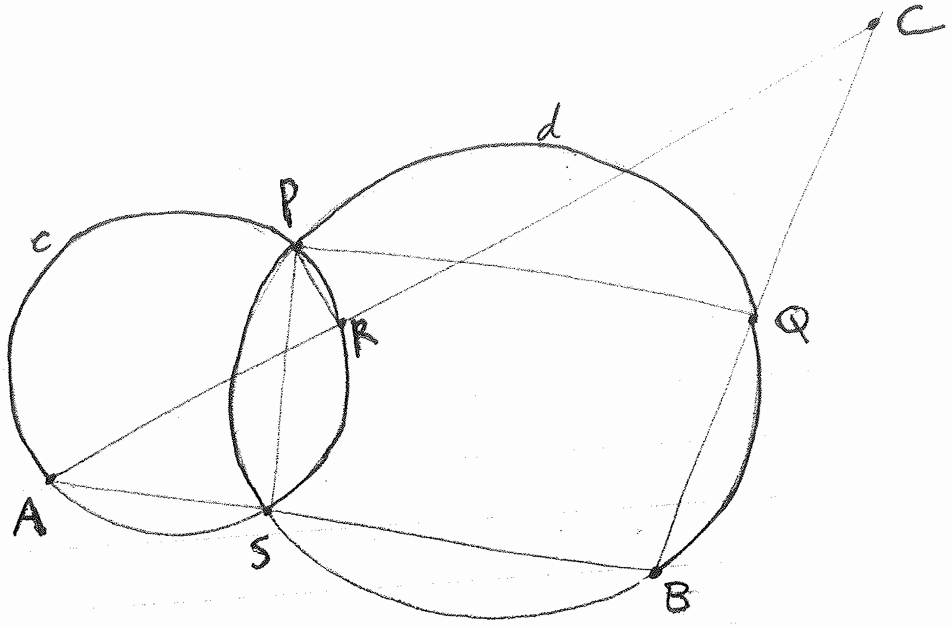
$$\therefore V = 28\pi \times \frac{1}{2} \times \pi \times 3^2$$

$$\therefore V = 126\pi^2 \text{ units}^3.$$

[ALT: the integral may be done from scratch, using a substitution $y = 3\sin\theta$.].

(c) (i)

(i)



(ii) $\angle PRA = \angle PSA$ (angles in same segment of circle c)

$= \angle PQB$ (exterior angle of cyclic quad. PQBS equals opposite interior angle)

✓✓

(iii) $\angle PRA = \angle PQB$ (from (ii))

$\therefore$ $180 - \angle PRA = 180 - \angle PQB$

i.e. $\angle PRC = \angle PQC$ (angles on straight line are supplementary)

$\therefore$ PRQC is cyclic (interval PC subtends equal angles at two points on the same side of it).

✓✓

Question 14: (a) $xy = c^2$ and $4ay = x^2$. $A(2at, at^2)$

(i) $y' = \frac{2x}{4a} = \frac{x}{2a}$ at $x = 2at$ $y' = t$ ✓

$y - at^2 = t(x - 2at)$

$y = tx - 2at^2 + at^2$

$\boxed{y = tx - at^2}$ (1) ✓

(ii) Tangent and hyperbola meet at B and C

Multiply (1) by x, we obtain

[alt: solve simultaneously].

$xy = tx^2 - at^2x$, but $xy = c^2$

$\therefore$ $\boxed{tx^2 - at^2x - c^2 = 0}$ (2) ✓

The solution of equation (2) are the x-coordinates of B and C. So the sum of roots equal at and the x-coordinate of M the midpoint of BC is $\frac{1}{2}at$ ✓

M lies on the tangent $\therefore$ M satisfy equation (1)

$y = t\left(\frac{1}{2}at\right) - at^2 = \frac{1}{2}at^2 - at^2 = -\frac{1}{2}at^2$

So $M\left(\frac{1}{2}at, -\frac{1}{2}at^2\right)$ ✓

iii) $x = \frac{1}{2}at \therefore t = \frac{2x}{a}$

$y = -\frac{1}{2}at^2 = -\frac{1}{2}a\left(\frac{2x}{a}\right)^2 = -\frac{1}{2}a \times \frac{4x^2}{a^2} = \frac{-2x^2}{a}$ ✓

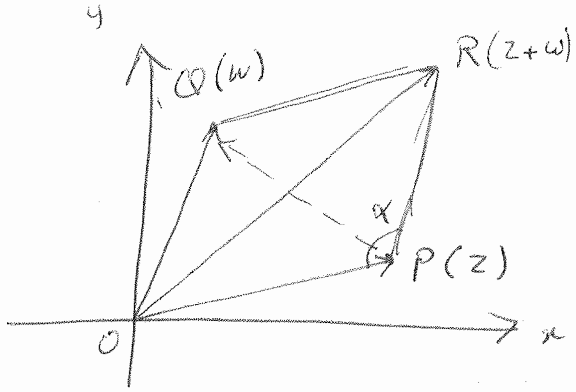
In Equation (2) the roots equal $\frac{at^2 \pm \sqrt{a^2t^4 + 4c^2t}}{2t}$

Since t takes all values $\therefore$ $a^2t^4 + 4c^2t$ can not be negative

$\therefore$ $a^2\left(\frac{2x}{a}\right)^4 + 4c^2\left(\frac{2x}{a}\right)$ can not be negative

$\frac{16x^4}{a^2} + \frac{8c^2}{a}x = \frac{8}{a^2}\left(2x^4 + ac^2x\right) = \frac{8}{a^2}x\left(2x^3 + ac^2\right) \geq 0$

$x < \left(\frac{-ac^2}{2}\right)^{1/3}$ or $x \geq 0$ ✓



(b)

(i) $\angle POQ = \arg(w) - \arg(z)$
$= \arg\left(\frac{w}{z}\right)$

OPRQ is a parallelogram

$\therefore$ $\angle POQ$ and $\angle OPR$ are supplementary

$\therefore$ $\arg\left(\frac{w}{z}\right) + \alpha = \pi$

and $\arg\left(\frac{w}{z}\right) = \pi - \alpha$

(ii) $\overrightarrow{OR}$ represents $z+w$

$$|z+w|^2 = |z|^2 + |w|^2 - 2|z||w|\cos\alpha$$
$$= |z|^2 + |w|^2 - 2|z||w|\cos\left(\pi - \arg\left(\frac{w}{z}\right)\right)$$
$$= |z|^2 + |w|^2 + 2|z||w|\cos\left(\arg\left(\frac{w}{z}\right)\right) \quad (1)$$

(iii) in $\triangle OPQ$, $\overrightarrow{PQ}$ represents $w-z$

$$|w-z|^2 = |w|^2 + |z|^2 - 2|w||z|\cos\left(\arg\left(\frac{w}{z}\right)\right) \quad (2)$$

$(1)-(2)$ $\therefore$ $|z+w|^2 - |w-z|^2 = 4|z||w|\cos\left(\arg\left(\frac{w}{z}\right)\right)$

$$\therefore \cos\left(\arg\left(\frac{w}{z}\right)\right) = \frac{|z+w|^2 - |w-z|^2}{4|z||w|}$$

(iv) $\alpha = \frac{7\pi}{12}$, $|PR| = 2$, $z = 3\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, $\arg(z) = \pi/4$

$$\arg\left(\frac{w}{z}\right) = \pi - \alpha = \pi - \frac{7\pi}{12} = \frac{5\pi}{12}$$
$$\arg(w) = \arg(z) + \frac{5\pi}{12} = \pi/4 + \frac{5\pi}{12} = \frac{2\pi}{3}$$

and $|w| = 2$ $\therefore$ $w = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$$w + z = 2\,\text{cis}\,\frac{2\pi}{3} + 3\,\text{cis}\,\frac{\pi}{4}$$
$$= 2\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right) + 3\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$$
$$= -1 + \frac{3}{\sqrt{2}} + i\left(\sqrt{3} + \frac{3}{\sqrt{2}}\right)$$

Question 15: $z = \text{cis}\,\theta$, $z^n - \frac{1}{z^n} = 2i\sin n\theta$

(i) $\left(z - \frac{1}{z}\right)^5 = z^5 - 5z^3 + 10z - \frac{1}{10}z + 5\frac{1}{z} - \frac{1}{z^5}$

$|z| = 1$, $\frac{1}{z} = \bar{z}$, $z - \bar{z} = 2i\sin\theta$

$$\left(z - \frac{1}{z}\right)^5 = (2i\sin\theta)^5 = z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$$2^5 i \sin^5\theta = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$$

$$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta) \quad (*)$$

(ii) $\sin 5\theta - 5\sin\theta + 6\sin\theta = 0$

but $\sin 5\theta - 5\sin\theta + 6\sin\theta = 16\sin^5\theta - 4\sin\theta$ from (*)

$\therefore 4\sin\theta(4\sin^4\theta - 1) = 0$

$\therefore 4\sin\theta(2\sin^2\theta - 1)(2\sin^2\theta + 1) = 0$

$\therefore \sin\theta = 0$ or $\sin\theta = \pm\frac{1}{\sqrt{2}}$

$\theta = 0, \pi, 2\pi, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $7\pi/4$.

(b) (i) $I_n = \int_0^{\pi/4} \sec^n x\,dx = \int_0^{\pi/4} \sec^{n-2}x \sec^2 x\,dx$

let $u = \sec^{n-2}x$ and $\sec^2 x\,dx = dv$

$du = (n-2)\sec^{n-3}x \times \sec x\tan x\,dx$ and $v = \tan x$

$$I_n = \left[\sec^{n-2}x\tan x\right]_0^{\pi/4} - (n-2)\int_0^{\pi/4}\sec^{n-2}x\tan^2 x\,dx$$

$$= (\sqrt{2})^{n-2} - (n-2)\int_0^{\pi/4}\sec^{n-2}x(\sec^2 x - 1)\,dx$$

$$= (\sqrt{2})^{n-2} - (n-2)\int_0^{\pi/4}\sec^n x\,dx + (n-2)\int_0^{\pi/4}\sec^{n-2}x\,dx$$

$$= (\sqrt{2})^{n-2} - (n-2)I_n + (n-2)I_{n-2}$$

$(n-1)I_n = (\sqrt{2})^{n-2} + (n-2)I_{n-2}$ $\therefore$ $\boxed{I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1}I_{n-2}}$

Q15

(b) (i)

$$I_n = \int_0^{\pi/4} \sec^n x \, dx$$

$$= \int_0^{\pi/4} \sec^2 x \sec^{n-2} x \, dx$$

$$= \int_0^{\pi/4} (\tan^2 x + 1) \sec^{n-2} x \, dx$$

$$= \underbrace{\int_0^{\pi/4} \tan^2 x \sec^{n-2} x \, dx} + \int_0^{\pi/4} \sec^{n-2} x \, dx$$

$u = \tan x \qquad dv = \tan x \sec^{n-2} x \, dx$

$du = \sec^2 x \, dx \qquad v = \dfrac{\sec^{n-2} x}{n-2}$

$$\therefore I_n = \left[\frac{\tan x \sec^{n-2} x}{n-2} \right]_0^{\pi/4} - \frac{1}{n-2} \int \sec^n x \, dx + I_{n-2}$$

$$= \frac{\sqrt{2}^{\,n-2}}{n-2} - \frac{1}{n-2} I_n + I_{n-2}$$

$$(n-2) I_n = \sqrt{2}^{\,n-2} - I_n + (n-2) I_{n-2}$$

$$(n-2+1) I_n = \sqrt{2}^{\,n-2} + (n-2) I_{n-2}$$

$$I_n = \frac{\sqrt{2}^{\,n-2}}{n-1} \quad \frac{n-2}{n-1} I_{n-2}$$

(ii) $\displaystyle\int_0^1 (1+x^2)^{5/2}\,dx$

let $x = \tan u \ \therefore\ dx = \sec^2 u\,du$

$(1+x^2)^{5/2} = (1+\tan^2 u)^{5/2} = (\sec^2 u)^{5/2} = \sec^5 u$

for $x = 0 \ \therefore\ u = 0$

$x = 1 \ \therefore\ u = \pi/4$

$$\int_0^1 (1+x^2)^{5/2}\,dx = \int_0^{\pi/4} \sec^5 u \sec^2 u\,du = \int_0^{\pi/4} \sec^7 u\,du = I_7$$

Use (i) $\displaystyle I_7 = \frac{(\sqrt{2})^5}{6} + \frac{5}{6} I_5$

$$I_5 = \frac{(\sqrt{2})^3}{4} + \frac{3}{4} I_3$$

$$I_3 = \frac{(\sqrt{2})}{2} + \frac{1}{2} I_1$$

$$I_1 = \int_0^{\pi/4} \sec x\,dx = \int_0^{\pi/4} \frac{\sec x\,(\sec x + \tan x)}{\sec x + \tan x}\,dx$$

$$= \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/4} = \ln(1+\sqrt{2})$$

$$I_3 = \frac{\sqrt{2}}{2} + \frac{\ln(1+\sqrt{2})}{2}$$

$$I_5 = \frac{\sqrt{2}}{2} + \frac{3}{4}\left(\frac{\sqrt{2}}{2} + \frac{\ln(1+\sqrt{2})}{2}\right) = \frac{7\sqrt{2}}{8} + \frac{3\ln(1+\sqrt{2})}{8}$$

$$I_7 = \frac{2\sqrt{2}}{3} + \frac{5}{6}\left(\frac{7\sqrt{2}}{8} + \frac{3\ln(1+\sqrt{2})}{8}\right)$$

$$= \frac{67}{48}\sqrt{2} + \frac{15}{48}\ln(1+\sqrt{2})$$

(c) $x^3 - 2bx^2 - a^2x + b^2 = 0$

(i) $x = -1$ is a solution $\therefore$

$(-1)^3 - 2b(-1)^2 - a^2(-1) + b^2 = 0$

$-1 - 2b + a^2 + b^2 = 0$

$a^2 = 1 + 2b - b^2 = 2 - (b+1)^2 \geqslant 0$ since $a^2 \geqslant 0$.

$\therefore \quad -\sqrt{2} \leqslant b - 1 \leqslant \sqrt{2}$

$\therefore \quad 1 - \sqrt{2} \leq b \leq 1 + \sqrt{2}$

(ii) $x = -1$ is a repeated root

$\therefore$ $x = -1$ is a solution of $x^3 - 2bx^2 - a^2x + b^2 = 0$

Also $x = -1$ is a solution of $3x^2 - 4bx - a^2 = 0$ (the derivative)

$\therefore$ $-1 - 2b + a^2 + b^2 = 0$ from (i)

and $3(-1)^2 - 4b(-1) - a^2 = 0$ $\therefore$ $3 + 4b - a^2 = 0$

Now $-1 - 2b + a^2 + b^2 = 0$ (1)

$3 + 4b - a^2 = 0$ (2)

(1) + (2) $\therefore$ $2 + 2b + b^2 = 0$

$(b+1)^2 + 1 = 0$ (3)

but b is a real number, hence there is no value of b which satisfy equation (3).

Question 16 :

(a) $x > 0$ real number

(i) $1 - t^3 + t^6 - t^9 + \dots + t^{6n}$

$2n+1$ terms of geometric series, their sum equal

$$\frac{1-(-t^3)^{2n+1}}{1-(-t^3)} = \frac{1+(t^3)^{2n+1}}{1+t^3}$$

(ii) $$1 - t^3 + t^6 - t^9 + \dots + t^{6n} = \frac{1+(t^3)^{2n+1}}{1+t^3} = \frac{1}{1+t^3} + \frac{(t^3)^{2n+1}}{1+t^3} > \frac{1}{1+t^3}$$

$$\therefore \boxed{\frac{1}{1+t^3} < 1 - t^3 + t^6 - \dots + t^{6n}} \quad (1)$$

(iii) $1 - t^3 + t^6 - t^9 + \dots + t^{6n} - t^{6n+3}$

$2n+2$ terms of geometric series, their sum equal

$$\frac{1-(-t^3)^{2n+2}}{1+t^3} = \frac{1-(t^3)^{2n+2}}{1+t^3}$$

(iv) $$1 - t^3 + t^6 - t^9 + \dots + t^{6n} - t^{6n+3} = \frac{1-(t^3)^{2n+2}}{1+t^3} = \frac{1}{1+t^3} - \frac{(t^2)^{2n+2}}{1+t^3} < \frac{1}{1+t^3}$$

$$\therefore 1 - t^3 + t^6 - t^9 + \dots + t^{6n} - t^{6n+3} < \frac{1}{1+t^3}$$

and $$\boxed{1 - t^3 + t^6 - t^9 + \dots + t^{6n} < \frac{1}{1+t^3} + t^{6n+3}} \quad (2)$$

(v) (1) and (2) $\therefore$

$$\frac{1}{1+t^3} < 1 - t^3 + t^6 - t^9 + \dots + t^{6n} < \frac{1}{1+t^3} + t^{6n+3} \quad (3)$$

Multiply (3) by t^2, we obtain

$$\frac{t^2}{1+t^3} < t^2 - t^5 + t^8 - t^{11} + \dots + t^{6n+2} < \frac{t^2}{1+t^3} + t^{6n+5} \quad (4)$$

$$\int_0^x \frac{t^2}{1+t^3}\,dt < \int_0^x \left(t^2 - t^5 + t^8 - t^{11} + \cdots + t^{6n+2}\right)dt < \int_0^x \frac{t^2}{1+t^3}\,dt + \int_0^x t^{6n+5}\,dt$$

$$\left[\frac{1}{3}\ln\left(1+t^3\right)\right]_0^x < \left[\frac{t^3}{3} - \frac{t^6}{6} + \frac{t^9}{9} - \frac{t^{12}}{12} + \cdots + \frac{t^{6n+3}}{6n+3}\right]_0^x < \left[\frac{1}{3}\ln\left(1+t^3\right)\right]_0^x + \left[\frac{t^{6n+6}}{6n+6}\right]_0^x$$

$$\frac{1}{3}\ln\left(1+x^3\right) < \frac{x^3}{3} - \frac{x^6}{6} + \frac{x^9}{9} - \frac{x^{12}}{12} + \cdots + \frac{x^{6n+3}}{6n+3} < \frac{1}{3}\ln\left(1+x^3\right) + \frac{x^{6n+6}}{6n+6}$$

(vi) As $n \longrightarrow \infty$

$$\frac{1}{3}\ln\left(1+x^3\right) < \frac{x^3}{3} - \frac{x^6}{6} + \cdots < \frac{1}{3}\ln\left(1+x^3\right) + 0$$

$$\therefore \quad \ln\left(1+x^3\right) = x^3 - \frac{x^6}{2} + \frac{x^9}{3} - \frac{x^{12}}{4} + \cdots$$

(vii) take $x = 1$

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

(viii) $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Combining 1 and $-\frac{1}{2}$, $\frac{1}{3}$ and $-\frac{1}{4}$ and so on, we obtain:

$$\ln 2 = \frac{1}{2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots$$

$$2\ln 2 = 1 + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{4\cdot 7} + \cdots$$

$$\ln 4 = 1 + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{4\cdot 7} + \cdots$$

$\square$

(b) (i) $\frac{1}{xy} + x + y \geqslant 3\sqrt[3]{\frac{1}{xy} \times x \times y} = 3$

(ii) $\frac{1}{a(a+1)} + \frac{1}{b(b+1)} + \frac{1}{c(c+1)} \geqslant 4$?

$\frac{1}{a(a+1)} + a + a+1 \geqslant 3$ (1) replacing x by a and y by $a+1$ in (i)

Similarly

$\frac{1}{b(b+1)} + b + b+1 \geqslant 3$ (2)

$\frac{1}{c(c+1)} + c + c+1 \geqslant 3$ (3)

(1) + (2) + (3) :

$\frac{1}{a(a+1)} + \frac{1}{b(b+1)} + \frac{1}{c(c+1)} + a+b+c + a+b+c + 3 \geqslant 3+3+3$

$\frac{1}{a(a+1)} + \frac{1}{b(b+1)} + \frac{1}{c(c+1)} + 1 + 1 + 3 \geqslant 9$ since $a+b+c=1$

$\frac{1}{a(a+1)} + \frac{1}{b(b+1)} + \frac{1}{c(c+1)} \geqslant 9-5 = 4$.

$\square$